Fixed Point Theorems and Related Topics in Abstract Convex Spaces (Mathematical Economics)

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Fixed Point Theorems and Related Topics in Abstract Convex Spaces

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It is well known that we can derive important results such as fixed point theorems, separation theorems, selection theorems of continuous functions, etc. from convexity in linear space structure. Many researchers have tried to extend these theorems under weaker conditions, apply them in general circumstances and get new results. We shall introduce the study of H-spaces which started with the concept of singular face structures due to Horvath[8]. We mainly cite the topics discussed below from [8], [9], [11].

Definition 1 Let X be a topological space and $\mathcal{F}$(X) be the family of all nonvoid finite subsets of X. A mapping $F : \mathcal{F}(X) \twoheadrightarrow X$ is said to be a *singular face structure* if it satisfies the following two conditions:

1. For any $A \in \mathcal{F}(X)$, $F(A)$ is nonvoid and contractible;
2. For any $A, B \in \mathcal{F}(X)$ with $A \subset B$, $F(A) \subset F(B)$.

Definition 2 A pair (X, F) of a topological space X and a singular face structure F is said to be an *H-space*. A subset C of X is said to be *convex* if $F(A) \subset C$ holds for any $A \in \mathcal{F}(C)$. An H-space (X, F) is said to be a *locally convex metric H-space* if X is a metric space, and a set $\{x \in X : d(x, A) < \epsilon\}$ is convex for any $\epsilon > 0$ and any convex set A, and any open ball is convex.

1 KKM Type Theorems and Fixed Point Theorems

The following proposition is fundamental to develop the theory of H-spaces.

Proposition 1 Let (X, F) be an H-space and $A \in \mathcal{F}(X)$ with $A = \{a_1, \ldots, a_n\}$. Denote by Δ_{n-1} the standard $(n-1)$ dimensional simplex co$\{e_1, \ldots, e_n\}$in R^n. Then there is a continuous function $f : \Delta_{n-1} \to X$ satisfying the following conditions:

$$f(\Delta_J) \subset F(A_J) \text{ for any } J \subset \{1, \ldots, n\},$$

where $\Delta_J = \text{co}\{e_j : j \in J\}$ and $A_J = \{a_j : j \in J\}$.

Proof Firstly, take $x_i \in F(\{a_i\})$ arbitrarily for each $a_i \in A$, and define a continuous function $f^0 : \Delta_{n-1}^0 \to X$ by $f^0(e_i) = x_i$, where Δ_{n-1}^0 denotes the 0-dimensional skeleton of Δ_{n-1}. Secondly, assume that, for any k dimensional skeleton Δ_{n-1}^k of Δ_{n-1}, there is a continuous function $f^k : \Delta_{n-1}^k \to X$ such that $f^k(\Delta_J) \subset F(A_J)$ for all J with $|J| \leq k+1$.

Let Δ_J be a $k+1$ dimensional face of Δ_{n-1}. Put $J_i = J \setminus \{i\}$ for $i \in J$. Since the boundary $\partial\Delta_J = \bigcup_{i \in J} \Delta_{J_i}$ of Δ_J is included in the k dimensional skeleton Δ_{n-1}^{k-1} of Δ_{n-1}, we have

$$f^k(\partial\Delta_J) \subset \bigcup_{i \in J} f^k(\Delta_{J_i}) \subset \bigcup_{i \in J} F(A_{J_i}) \subset F(A_J).$$

Since $F(A_J)$ is contractible, f^k can be extended to a continuous function $f_J^{k+1} : \Delta_J \to F(A_J)$ on Δ_J. Let Δ_J and $\Delta_{J'}$ be two $k+1$ dimensional faces of Δ_{n-1}such that $\Delta_J \cap \Delta_{J'} \neq \emptyset$. Then f_J^{k+1} and $f_{J'}^{k+1}$ have the same values as f^k on $\Delta_J \cap \Delta_{J'}$ since $\Delta_J \cap \Delta_{J'}$ is contained in the k dimensional skeleton Δ_{n-1}^k. Hence, we can paste all of the continuous functions on k+1 dimensional faces constructed above and make a continuous function $f^{k+1} : \Delta_{n-1}^{k+1} \to R$ on the $(k+1)$ dimensional skeleton. Repeating this process, we have a desired continuous function f on Δ_{n-1}. $\square$

The next theorem is an H-space version of the KKM theorem.

Theorem 1 Let (X, F) be an H-space. Let $A \in \mathcal{F}(X)$ with $A = \{a_1, \ldots, a_n\}$ and $\{R_i\}_{i=1}^n$ be a family of closed subsets of X. If $F(A_J) \subset \bigcup_{i \in J} R_i$ holds for any $J \subset \{1, \ldots, n\}$, then we have $\bigcap_{i=1}^n R_i \neq \emptyset$.

Proof Let $f : \Delta_{n-1} \to X$ be a continuous function obtained in Proposition 1. Then $f(\Delta_J) \subset F(A_J) \subset \bigcup_{i \in J} R_j$ holds for any J. Hence, it follows that $\Delta_J \subset f^{-1}(\bigcup_{i \in J} R_i) = \bigcup_{i \in J} f^{-1}(R_i)$, and there is a point $t_0 \in \Delta_{n-1}$ such that $t_0 \in \bigcap_{i=1}^n f^{-1}(R_i)$ by virtue of the KKM theorem. The point $f(t_0)$ belongs to $\bigcap_{i=1}^n R_i$. $\square$

The next is an H-space version of the extension of KKM theorem by Fan[6].

Theorem 2 Let (X, F) be a compact H-space, Y a subset of X. If a closed multi-valued mapping $\gamma : Y \twoheadrightarrow X$ satisfies $F(A) \subset \bigcup_{x\in A} \gamma(x)$ for any $A \in \mathcal{F}(Y)$, then it follows that $\bigcap_{x\in Y} \gamma(x) \neq \emptyset$.

Proof We have $\bigcap_{x\in Y} \gamma(x) \neq \emptyset$ because $\{\gamma(x)\}_{x\in Y}$ has the finite intersection property by virtue of Theorem 1. $\square$

Definition 3 For a multi-valued mapping $\gamma : X \twoheadrightarrow Y$, we define the *dual* $\gamma^* : Y \twoheadrightarrow X$ of γ by $\gamma^*(y) = X \setminus \gamma^{-1}(y)$.

Definition 4 Given multi-valued mapping $\gamma : X \twoheadrightarrow X$, a point $x_0 \in X$ with $\gamma(x_0) = \emptyset$ is said to be a *maximal element* of γ, and a point of $\bigcap_{x\in X} \gamma(x)$ is said to be a *maximum element* of γ. From the definitions above, it is easily seen that a maximal element of γ is a maximum element of γ^*, and a maximum element of γ is a maximal element of γ^*. Moreover, it follows $\gamma = \gamma^{**}$.

The existence of a maximum element is nothing but the conclusion of a KKM type theorem, and hence we can easily prove the following theorem using a KKM type theorem.

Theorem 3 Let (X, F) be a compact H-space and suppose that $\gamma : X \twoheadrightarrow X$ enjoys the following properties:

1. $\gamma(x)$ is closed for any $x \in X$;
2. $x \in \gamma(x)$ holds for any $x \in X$;
3. $\gamma^*(x)$ is convex for any $x \in X$.

Then, we have $\bigcap_{x\in X} \gamma(x) \neq \emptyset$.

Proof We only need to show $F(A) \subset \bigcup_{a\in A} \gamma(a)$ for any $A \in \mathcal{F}(X)$ by virtue of Theorem 2. If there is a point x such that $x \in F(A) \setminus \bigcup_{a\in A} \gamma(a)$, then $a \in \gamma^*(x)$ for any $a \in A$, and hence, we have $A \subset \gamma^*(x)$. Since $\gamma^*(x)$ is convex by the assumption, we have $F(A) \subset \gamma^*(x)$. Therefore, it follows $x \in F(A) \subset \gamma^*(x)$, and we have $x \notin \gamma(x)$, but this contradicts our assumption. $\square$

We have the following theorem paying attention to Theorem 3 above and the duality between maximal elements and maximum elements.

Theorem 4 Let (X, F) be a compact H-space, and suppose that $\varphi : X \twoheadrightarrow X$ satisfies the following conditions:

1. $\varphi(x)$ is convex for any $x \in X$;
2. $x \notin \varphi(x)$ for any $x \in X$;
3. $\varphi^{-1}(x)$ is open for any $x \in X$.

Then there is $x \in X$ such that $\varphi(x) = \emptyset$.

Proof If we put $\gamma = \varphi^*$, then γ satisfies all of the hypotheses of Theorem 3. Hence there is $y \in X$ such that $y \in \bigcap_{x \in X} \gamma(x)$, that is, we have $x \notin \varphi(y)$ for all $x \in X$, and $\varphi(y) = \emptyset$. □

2 Continuous Selections

Theorem 5 Let X be a paracompact topological space, and (Y, F) an H-space. Suppose that a multi-valued mapping $\varphi : X \twoheadrightarrow Y$ enjoys the following properties:

1. $\varphi(x)$ is nonvoid and convex for all $x \in X$;
2. $\varphi^{-1}(y)$ is open for all $y \in X$.

Then, φ has a continuous selection.

Proof Note that the family $\{\varphi^{-1}(y)\}_{y \in Y}$ is an open covering of X. Since X is paracompact, there are an locally finite open covering $\mathcal{U} = \{U\}$ of X and a function $y : \mathcal{U} \to Y$ such that $U \subset \varphi^{-1}(y(U))$ for all $U \in \mathcal{U}$. Let $\{\beta_U\}$ be a partition of unity subordinate to $\mathcal{U}$. Let $\mathcal{N}$ be the nerve of the locally finite open covering $\mathcal{U}$, and let $|\mathcal{N}|$ be the geometrical realization of the nerve $\mathcal{N}$. Denote by $v(U)$ the vertex of $|\mathcal{N}|$ corresponding to an open set U in $\mathcal{U}$. Now define a continuous function $f : X \to |\mathcal{N}|$ by

$$f(x) = \sum_U \beta_U(x) v(U), \quad x \in X.$$

On the other hand, define a function $\eta : |\mathcal{N}|^0 \to Y$ by $\eta = y \circ v^{-1}$, where $|\mathcal{N}|^0$ is the 0 dimensional skeleton of $|\mathcal{N}|$. Denote a simplex in $|\mathcal{N}|$ by s and the set of all vertexes of s by s^0. We consider a pair (L, g), where L is a sub-complex of $|\mathcal{N}|$ and g is a function on L to Y such that $g(s) \subset F(\eta(s^0))$ for all simplex s in L. Let Z be the set of all the pairs of this type. Moreover, define a partial order $\leq$ in Z by $(L, g) \leq (L', g')$ if and only if L is a sub-complex of L' and $g'|_L = g$.

Firstly, if we define a function $g : |\mathcal{N}|^0 \to Y$, where $|\mathcal{N}|^0$ is the 0 dimensional skeleton of $|\mathcal{N}|$, by $g(u) \in F(\eta(u))$ for each vertex u, then $(|\mathcal{N}|^0, g)$ is an element of Z and we can conclude Z is nonvoid.

Secondly, take a chain $C = \{(L_i, g_i)\}_{i \in I}$ in $(Z, \leq)$. Define $\overline{L}$ by $\overline{L} = \bigcup_{i \in I} L_i$, and $\overline{g} : \overline{L} \to Y$ by $\overline{g}(x) = g_i(x)$ for x with $x \in L_i$. Then $(\overline{L}, \overline{g})$ belongs to Z, and it is obviously an upper bound of C. Therefore, $(Z, \leq)$ has a maximal element (L, g) by Zorn's lemma. We can establish the equation $L = |\mathcal{N}|$ as follows.

If $L \neq |\mathcal{N}|$, then there is a k dimensional skeleton of $|\mathcal{N}|$ which is not contained in L. Let k_0 be the minimum value of such k's. It is impossible $k_0 = 0$. Indeed, if $k_0 = 0$, then there is a vertex $u \notin L$, but we can extend g to $L \cup \{u\}$ and this contradicts the maximality of (L, g). Let s be a k_0-simplex of $|\mathcal{N}|$ not belonging to L. The boundary ∂s of s is contained in the $(k_0 - 1)$ dimensional skeleton of $|\mathcal{N}|$, and hence in L. If t is one of the faces of s constructing ∂s, then $g(t) \subset F(\eta(t^0)) \subset F(\eta(s^0))$, and hence we have $g(\partial s) \subset F(\eta(s^0))$. However, since $F(\eta(s^0))$ is contractible, we can extend g to s and obtain the extension $g' : s \to F(\eta(s^0))$. If we put $\widetilde{L} = L \cup s$, $\widetilde{L}$ becomes a sub-complex of $|\mathcal{N}|$ because $\partial s \subset L$. Moreover, define $\widetilde{g} : \widetilde{L} \to Y$ such that $\widetilde{g}$ is equal to g on L and is equal to g' on s. Then $(\widetilde{L}, \widetilde{g})$ belongs to Z, and this contradicts the maximality of (L, g) and we have a contradiction. Now we have proved the existence of a continuous function $g : |\mathcal{N}| \to Y$ such that $g(s) \subset F(\eta(s^0))$ for all simplexes s in $|\mathcal{N}|$.

Take the composite $g \circ f$ of $f : X \to |\mathcal{N}|$, which is constructed in the first part of this proof, and $g : |\mathcal{N}| \to Y$ whose existence we have just proved. Then this is a continuous selection of φ. Indeed, take any point $x \in X$. Let s be the simplex in $|\mathcal{N}|$ whose vertexes are $\{v(U) : U \ni x\}$. Then we have

$$g(f(x)) = g(\sum_U \beta_U(x) v(U)) \in g(s) \subset$$

$$F(\{\eta(v(U)) : U \ni x\}) \subset F(\{y(U) : U \ni x\}) \subset \varphi(x).$$

The last inclusion is verified as follows: If $x \in U$, then we have $x \in \varphi^{-1}(y(U))$ by the definition of y, and $F(\{y(U) : U \ni x\}) \subset \varphi(x)$ by the fact $y(U) \in \varphi(x)$

and the convexity of $\varphi(x)$. □

Proposition 2 Let X be a paracompact topological space, (Y, F) a locally convex metric H-space. Suppose that a multi-valued mapping $\varphi : X \twoheadrightarrow Y$ enjoys the following properties:

1. φ is lower semicontinuous;

2. $\varphi(x)$ is nonvoid convex for all $x \in X$.

Then, for any $\epsilon > 0$, there is a continuous function $g : X \to Y$ with the following property:

$$\varphi(x) \cap B(g(x), \epsilon) \neq \emptyset \text{ for all } x \in X.$$

Moreover, the multi-valued mapping $\varphi' : X \twoheadrightarrow Y$ defined by $\varphi'(x) = \varphi(x) \cap B(g(x), \epsilon)$ is lower semicontinuous.

Proof Define $\psi : X \twoheadrightarrow Y$ by $\psi(x) = \{y \in Y : \varphi(x) \cap B(y, \epsilon) \neq \emptyset\}$. Then, $\psi(x)$ is nonvoid and convex, and $\psi^{-1}(y) = \{x \in X : \varphi(x) \cap B(y, \epsilon) \neq \emptyset\}$ is open because $B(y, \epsilon)$ is open and φ is lower semicontinuous. ψ has a continuous selection g by Theorem 5, and this g is the desired function.

The lower semicontinuity of φ' is shown as follows. If we put $B = \{(y, y') \in Y \times Y : d(y, y') < \epsilon\}$, then we have the equation

$$\{x \in X : G \cap \varphi'(x) \neq \emptyset\} = \{x \in X : (\{g(x)\} \times \varphi(x)) \cap (B \cap (Y \times G)) \neq \emptyset\}$$

for any subset G of Y, and the multi-valued mapping $x \mapsto \{g(x)\} \times \varphi(x)$ from X to $Y \times Y$ is lower semicontinuous. □

Theorem 6 Let X be a paracompact topological space, (Y, F) a locally convex complete metric H-space. Suppose that a multi-valued mapping $\varphi : X \twoheadrightarrow Y$ enjoys the following properties:

1. φ is lower semicontinuous;

2. $\varphi(x)$ is nonvoid closed convex for any $x \in X$.

Then, φ has a continuous selection.

Proof By virtue of Proposition 2 and the mathematical induction on n, we can find a sequence $\{f_n\}$ of continuous functions satisfying $\varphi(x) \cap \bigcap_{k=1}^{n} B(f_n(x), 1/2^n) \neq \emptyset$. Since $d(f_{n+1}(x), f_n(x)) < 1/2^{n+1} + 1/2^n$ holds, $\{f_n\}$ is a Cauchy sequence in the sense of uniform convergence. Hence, f_n converges a continuous function f uniformly. It is easily seen that $f(x) \in \varphi(x)$ form the property of f_n. □

3 Fixed Point Properties

Definition 5 An H-space (X, F) is said to have the *fixed point property* if any continuous function from X into X has a fixed point. A multi-valued mapping γ from (X, F) into itself is said to be a *K-mapping* if it has nonvoid closed convex values and is upper semicontinuous. A multi-valued mapping φ from (X, F) into itself is said to be an *FB-mapping* if it has nonvoid convex values and the set $\varphi^{-1}(y) = \{x \in X : \varphi(x) \ni y\}$ is open for any $y \in X$. An H-space (X, F) is said to have the *K-fixed point property* if any K-mapping on (X, F) into itself has a fixed point, and have the *FB-fixed point property* if any FB-mapping on (X, F) into itself has a fixed point.

Proposition 3 Let (X, F) be a paracompact H-space. If (X, F) has the fixed point property, then (X, F) has the FB-fixed point property.

Proof The proof is trivial by Theorem 5. □

Proposition 4 Let (X, F) be a locally convex metric H-space. If (X, F) has the FB-fixed point property, then (X, F) has the K-fixed point property.

Proof We derive a contradiction assuming that (X, F) lacks the K-fixed point property. Let $\gamma : X \twoheadrightarrow X$ be a K-mapping with no fixed point. Since γ is upper semicontinuous and $\gamma(x)$ is closed, for any $x \in X$, there are an open neighborhood U_x of x and an open convex neighborhood G_x of $\gamma(x)$ such that $U_x \cap G_x = \emptyset$ and $\gamma(z) \subset G_x$ for all $z \in U_x$.

Since X is paracompact, there are a locally finite closed covering $\mathcal{V} = \{V\}$ of X and a function $x : \mathcal{V} \to X$ such that $V \subset U_{x(V)}$ for all $V \in \mathcal{V}$. For each $x \in X$, let W_x be a neighborhood of x such that the set $\{ V \in \mathcal{V} \mid V \cap W_x \neq \emptyset \}$ is a finite set.

Let $I_x = \{ V \in \mathcal{V} \mid V \ni x \}$ and $J_x = \{ V \in \mathcal{V} \mid V \cap W_x \neq \emptyset \}$. Define a multi-valued mapping $\varphi : X \twoheadrightarrow X$ by

$$\varphi(x) = \bigcap_{V \in I_x} G_{x(V)}, \qquad x \in X.$$

It is obvious that $\varphi(x)$ is convex for all $x \in X$. If $V \in I_x$, then $x \in V \subset U_{x(V)}$, and hence $\gamma(x) \subset G_{x(V)}$. Then, we have $\gamma(x) \subset \varphi(x)$, and φ is nonvoid-valued. On the other hand, it follows that $x \notin \varphi(x)$ for all $x \in X$ by the definition of φ.

Next we show that $\varphi^{-1}(y)$ is open for all $y \in X$. Let $x_0 \in \varphi^{-1}(y)$, that is , $y \in \bigcap_{V \in I_{x_0}} G_{x(V)}$. Define a set W by

$$W = \begin{cases} W_{x_0}, & \text{if } I_{x_0} = J_{x_0}; \\ W_{x_0} \cap \bigcap \{ V^c \mid V \in J_{x_0} \setminus I_{x_0} \} & \text{if } I_{x_0} \neq J_{x_0}. \end{cases}$$

Then, we have W is open and $x_0 \in W$. It is obvious that $I_z \subset I_{x_0}$ for all $z \in W$. Hence, it follows that

$$\varphi(z) = \bigcap_{V \in I_z} G_{x(V)} \supset \bigcap_{V \in I_{x_0}} G_{x(V)} = \varphi(x_0) \ni y.$$

Therefore, we have $\varphi^{-1}(y) \ni z$, that is, $\varphi^{-1}(y) \supset W$.

Summing up, φ is an FB-mapping, but $x \notin \varphi(x)$ for all $x \in X$, and φ has no fixed point. This contradicts the FB-fixed point property of X. □

We obtain the following theorem combining the previous two propositions.

Theorem 7 Let (X, F) be a locally convex metric H-space whose singleton sets are convex. Then, the following three statements are equivalent each other:

1. (X, F) has the fixed point property;
2. (X, F) has the FB-fixed point property;
3. (X, F) has the K-fixed point property.

We can regard a convex subset X of a locally convex metrizable linear topological space as a locally convex metric H-space (X, F) if we regard the mapping F as the usual operation of taking convex hulls in a linear space. Therefore, we have the following corollary to Theorem 7.

Crollary 1 Let X be a convex subset of a locally convex metrizable linear topological space. Then the following three statements are equivalent each other:

1. X has the fixed point property;
2. X has the FB-fixed point property;

3. X has the K-fixed point property.

Any non-compact convex subset of a locally convex metrizable linear topological space lacks the fixed point property by [10, Theorem 2.3], and hence we have the following corollary.

Crollary 2 Any non-compact convex subset of a locally convex metrizable linear topological space lacks the FB-fixed point property.

4 Topological Semilattices as H-spaces

We introduce topological semilattices with some conditions as nontrivial H-spaces.

Definition 6 A partially ordered set X with a topology is called a *topological semilattice* if there is a supremum $x \vee y$ for any two points $x, y \in X$, and the mapping $(x, y) \mapsto x \vee y$ from $X \times X$ to X is continuous.

Definition 7 A topological space X is said to be *ω-connected* if we can extend any continuous function from the n dimensional unit sphere S^n into X to a continuous function from the $n+1$ dimensional unit ball B^{n+1} into X for any natural number n.

Remark 1 A topological space X is ω-connected if and only if X is path-connected and the fundamental group $\pi_n(X, x)$ is trivial for any $x \in X$ and any natural number n. (cf. [15, page 51])

If a topological space X is contractible, then it is ω-connected, but the inverse is not necessarily true.

The discussion in the previous sections holds true even if we adopt the assumption that the values $F(A)$ of the singular face structure F are ω-connected instead of the assumption that the values $F(A)$ are contractible. We proceed in this section under the condition that the values $F(A)$ are ω-connected and we call a topological space an H-space if it has a singular face structure with ω-connected values.

Definition 8 Let X be a topological semilattice. Define $F : \mathcal{F}(X) \twoheadrightarrow X$ by

$$F(A) = \bigcup_{a \in A} [a, \sup A], \quad A \in \mathcal{F}(X).$$

If all of the order intervals in X are path-connected, then $F(A)$ is path-connected, and hence $F(A)$ is ω-connected by [4, Theorem B]. Therefore, (X, F) is an H-space. In this section, (X, F) denotes the pair of a topological semilattice X whose order intervals are all path-connected and a singular face structure F defined above.

The following proposition is trivial which describe the condition that a set is convex.

Proposition 5 A subset C of X is convex if and only if the following two conditions are true.

1. $x_1, x_2 \in C$ implies $x_1 \vee x_2 \in C$,
2. $x_1, x_2 \in C$ and $x_1 \leq x_2$ imply $[x_1, x_2] \subset C$.

Every proposition in the previous sections is true in a topological semilattice (X, F). We take Theorem 4 as an example, and interpret it in this context.

Theorem 8 Suppose that X is compact, and $\varphi : X \twoheadrightarrow X$ satisfies the following conditions:

1. If $x_1, x_2 \in \gamma(x)$, then $x_1 \vee x_2 \in \varphi(x)$;
2. If $x_1, x_2 \in \gamma(x)$, $x_1 \leq x_2$, then $[x_1, x_2] \subset \varphi(x)$;
3. $x \notin \varphi(x)$ for all $x \in X$;
4. $\varphi^{-1}(x)$ is open for all $x \in X$.

Then there is $x \in X$ such that $\varphi(x) = \emptyset$.

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